Int. J. Heat Mass Transfer. Vol. 8, pp. 1481–1490. Pergamon Press 1965. Printed in Great Britain

HEAT TRANSFER AND SKIN FRICTION IN A CHANNEL-LAMINAR FLOW WITH VARIABLE PHYSICAL PROPERTIES

V. P. TYAGI

Department of Mathematics, Indian Institute of Technology, Kanpur, India

(*Received* 19 *May* 1964 *and in revised form* 19 *February* 1965)

Abstract—This paper discusses the forced convection problem for the steady laminar fully developed flow in a straight channel when the fluid properties are functions of temperature. The energy equation has been analysed by taking into account the effect of compression work as well as that of viscous dissipation. The complex variable technique is used to tackle the case of an arbitrary channel. The constant property and variable property flows characteristics have been compared. For illustration some constant property flows have been obtained in a closed form and the results investigated by taking into account the effect of compression work have been compared with those obtained by neglecting it. For an arbitrary cross-section, the expressions giving the difference of the wall temperature from the initial temperature and the temperature drop in the duct, have been derived. In the concluding section, the results have been discussed.

NOMENCLATURE

a, some physical length measure defined in equations (30) and (21);

a_M, constant coefficients occurring in equation (16);

A, area of cross-section of a given duct which is constant in the axial direction;

b, semi-minor axis of an ellipse defined in equation (30);

B, bounding curve of the cross-section of a given duct;

$B_v(\frac{1}{2}, M+1)$, incomplete beta function, $\int_0^v (1 - v)^M v^{-\frac{1}{2}} \, dv$;

c_p, specific heat at a constant pressure referred to weight;

c_1, parameter defined in equation (8b);

D, region enclosed by B;

f_μ, known function of temperature introduced in equation (3a);

f_k, known function of temperature introduced in equation (3b);

$f(Z)$, function of a complex variable Z defined in equation (13);

g, acceleration due to gravity;

G, mass velocity;

i, $(-1)^{\frac{1}{2}}$;

K, coefficient of thermal conductivity;

l, physical length measure defined in equation (40);

p, pressure;

Pr, Prandtl number;

q, heat-transfer rate at the solid boundary;

r, radius vector defined in equation (40);

S, length measure of B;

t, local temperature;

T, $t - t_w$;

w, local velocity in the axial direction;

$\langle w \rangle$, mass flow rate;

x, y, z, Cartesian co-ordinates, z-co-ordinate is parallel to the axis of a given channel;

Z, complex variable, $x + iy$;

α_n, known complex constant coefficients introduced in equation (11);

γ, angle of inclination of the outward drawn normal through the current point of B with the axis of x;

σ, vectorial angle defined in equation (40);

θ, argument of ζ;

ϕ, eccentric angle introduced in equation (62);
ξ, boundary value of ζ;
ζ, complex variable in the mapped plane;
$\Omega(\zeta)$, function of ζ: mapping function;
ρ, density;
μ, coefficient of viscosity;
τ, skin friction;
$\hat{\tau}$, dimensionless skin friction, $\tau/(-L\,\mathrm{d}p/\mathrm{d}z)$;
τ^{+}, dimensionless skin friction, $\tau/(G^2/\rho)$;
∇^2, Laplacian operator, $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

Superscripts
prime, denotes differentiation with respect to the argument unless the contrary is specified;
bar, denotes conjugate complex of a quantity, e.g. $\bar{Z} = x - iy$.

Subscripts
d the value when the effect of pressure drop is neglected;
ex, extreme conditions within the channel;
l, local value;
m, average over the cross-section of a given duct;
o, initial condition;
w, condition at the solid boundary.

1. INTRODUCTION

In 1963, Riley [9] investigated the thermal boundary layer in a converging constant property flow between non-parallel plane walls without neglecting the important term representing the pressure contribution in the energy equation. Recently Madejski [7] has studied the combined effect of the pressure drop and the dissipation terms on the temperature field in the steady laminar fully developed constant property flow in straight channels. He investigated completely the cases of round and flat conduits with uniform wall temperature.

In the present paper we shall discuss an arbitrary variable property flow in a straight channel of any cross-section. The special case, in which viscosity and thermal conductivity vary with temperature in the same manner, will then be deduced directly. It will be assumed that the velocity and temperature fields are steady, laminar, and fully developed, the temperature differences are principally due to forced convection, and the fluid properties are temperature dependent [2]. The case of constant property flow will also be deduced directly from the variable property case, and then some specific examples will be analysed in order to compare the solution for constant properties which includes the contribution of compression work in the energy equation with the solution which neglects it.

2. THE MATHEMATICAL EQUATIONS GOVERNING THE PROBLEM

From the discussions of pages 124–127 of reference [8] and page 42 of reference [3] it is quite clear that, under certain circumstances, it becomes necessary to take into account the effect of compression work. In fact for all fluids to which the perfect gas law is applied, it is essential to consider the term involving the total time derivative of pressure in the energy equation even in incompressible motion. Thus the governing equations for the fluid flow (obeying the perfect gas law), under desired conditions (stated in Section 1), in any straight channel with uniform wall temperature, after [7], are:

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) \qquad (1)$$

$$0 = \frac{\partial}{\partial x}\left[\mu\frac{\partial}{\partial x}\left(\frac{w^2}{2}\right) + K\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\frac{\partial}{\partial y}\left(\frac{w^2}{2}\right) + K\frac{\partial T}{\partial y}\right] \qquad (2)$$

In deriving equation (2), use has been made of equation (1). Accordingly equation (2) contains the contribution of pressure drop also.

It is assumed that the rate of fall of pressure in the axial direction can be obtained from experimental data and is thus a known constant quantity.

Because of the temperature dependent viscosity, the velocity and temperature fields interact intimately and therefore, we have to consider momentum and energy equations simultaneously. We can use convenient semi-empirical relations to describe the temperature dependence of viscosity and thermal conductivity (e.g. the power law relations, the Sutherland's Law [12], etc.). For the present, however, we assume the general functional relationships:

$$\mu = f_\mu(T), \quad K = f_k(T). \qquad (3a, 3b)$$

The boundary conditions are

$$w = 0 \text{ on } B, \quad T = 0 \text{ on } B. \qquad (4a, 4b)$$

3. TRANSFORMS OF MOMENTUM AND ENERGY EQUATIONS

On some manipulation, it is found that the introduction of the quantities,

$$w_1 = \tfrac{1}{2}w^2, \qquad W = \int_0^{w_1} (\mu/\mu_w)\, \mathrm{d}w_1 \quad (5a, 5b)$$

$$T_1 = \int_0^T (K/K_w)\, \mathrm{d}T, \quad w_2 = \int_0^w (\mu/\mu_w)\, \mathrm{d}w \; (6a, 6b)$$

reduces equations (2) and (1) respectively to

$$\nabla^2\psi = 0, \qquad \psi = \mu_w W + K_w T_1 \quad (7a, 7b)$$

$$\nabla^2 w_2 = c_1, \qquad c_1 = \frac{1}{\mu_w} \cdot \frac{\mathrm{d}p}{\mathrm{d}z}. \quad (8a, 8b)$$

The reference quantities, μ_w and K_w, are constant under the assumption of uniform wall temperature.

The boundary conditions (4), obviously transform to:

$$\psi = 0 \text{ on } B, \quad w_2 = 0 \text{ on } B. \quad (9a, 9b)$$

4. SOLUTIONS FOR ψ AND w_2

For any boundary B, the only solution of (7) under the boundary condition (9a) is

$$\mu_w W + K_w \,.\, T_1 = 0. \qquad (10)$$

For a given boundary B, the solution of (8) under the boundary condition (9a) is most easily obtained by means of the technique of conformal mapping. If $Z = \Omega(\zeta)$ be the conformal map which transforms the region D on to the unit circle, $|\zeta| \leqslant 1$ in the ζ-plane, then $\Omega(\zeta)$ is also expressible [4] in the form

$$Z = \Omega(\zeta) = \sum_0 a_n \zeta^n \qquad (11)$$

and as a consequence the solution of (8), after [10], is

$$w_2 = c_1 \Big(\sum_0 a_n \zeta^n \sum_0 \bar{a}_n \bar{\zeta}^n - \sum_0 b_n \zeta^n - \sum_1 \bar{b}_n \bar{\zeta}^n\Big)/4, \quad b_n = \sum_0 a_{n+j}\, \bar{a}_j. \quad (12a, 12b)$$

However, for a certain class of boundaries, viz. those for which the equation can be expressed in the form

$$Z\bar{Z} = f(Z) + \bar{f}(\bar{Z}) \qquad (13)$$

the solution can be obtained in a simple closed form without employing the conformal transformation technique. In fact the solution is [5, 6]

$$w_2 = c_1[Z\bar{Z} - f(Z) - \bar{f}(\bar{Z})]/4. \qquad (14)$$

For illustration, (12) will be used to obtain the case of a Cardioid duct, and (14) will be employed to determine the cases of an equilateral triangular duct and an elliptic tube. The cases of circular and flat conduits will be shown deducible from that of an elliptic tube.

5. ANALYSIS

In general, when the functions (3) are arbitrary, the temperature and velocity profiles can be determined as follows: from (3b), substitute for K in (6a) in order to evaluate T_1. When this is substituted in (10), we get W in terms of T. With the aid of this, we can evaluate w_1, in terms of T from (5*b*) after substituting for μ from (3a) in (5b). Thus w is evaluated in terms of T from (5a) and then one can determine w_2 in terms of T with the aid of (6b) and (3a) easily. This, on using (12) or (14), gives us T in terms of x and y.

Although, as discussed above, it is always possible in principle to investigate temperature and velocity profiles, it is noteworthy that the arbitrary functions (3) may introduce considerable complications in the analysis. Hence for the sake of simplicity, we propose to discuss the case when viscosity and thermal conductivity have the same kind of temperature dependence [11, 12]. In this case the solution is

$$t = t_w - (Pr\, w^2)/(2g\, c_p). \qquad (15)$$

Here, the solution (15) is deduced directly from the general solution (10) after considering (5) and (6a).

As w_2 is already known in terms of space co-ordinates from (12) or (14), the velocity w and hence the temperature t from (15) can be obtained in terms of space co-ordinates after evaluating w_2 in terms of w from (6b). To achieve this, we require μ as a function of temperature. However, in general, one can write [1]

$$\mu = \sum_0 a_M t^M. \tag{16}$$

The coefficients a_M which influence the flow field are obtained from experimental data. The number of terms retained in equation (16) will be related to the accuracy of fit to experimental data. For a simple power law, for example, only one term is sufficient and the first term is retained for a constant property flow.

Eliminating t with the aid of (15), we find

$$\frac{\mu}{\mu_w} = (1/\sum_0 a_M t_w^M) \sum_0 a_M t_w^M [1 - (Pr\, w^2/2g\, c_p\, t_w)]^M. \tag{17}$$

Substitution of this in (6b) gives

$$w_2 = (1/\sum_0 a_M t_w^M) \sum_0 a_M t_w^M \sqrt{(g\, c_p\, t_w/2Pr)}\; B_v(\tfrac{1}{2}, M+1),\; v = (Pr\, w^2/2g\, c_p\, t_w). \tag{18a, 18b}$$

The velocity profile, w, in the variable property flow, in which viscosity and thermal conductivity vary with temperature in the same manner, can thus be obtained by eliminating w_2, either between (18a) and (12), or between (18a) and (14) whichever is convenient. In fact after doing this, it is not possible to evaluate w in the exact closed form in terms of space co-ordinates, nevertheless it is possible to compute it numerically in particular cases by employing, e.g. the mathematical tables of "Incomplete Beta Function".

As a special case, in constant property flow, w is obtainable in the exact closed form, because in this case $w_2 = w$ which is deducible from (6b) or (18). Thus from the relation (15) which holds good for the constant property flow as well, we have for B's defined by (13):

$$t = t_w - (c_1^2\, Pr/32g\, c_p)\, [Z\bar{Z} - f(Z) - \bar{f}(\bar{Z})]^2 \tag{19a}$$

and for an arbitrary B:

$$t = t_w - (c_1^2\, Pr/32g\, c_p)\, [\sum_0 a_n \zeta^n \sum_0 \bar{a}_n \bar{\zeta}^n - \sum_0 b_n \zeta^n - \sum_1 \bar{b}_n \bar{\zeta}^n]^2. \tag{19b}$$

For any given B, the expressions for the mass flow rate $\langle w \rangle = \rho\, w_m A = \rho \int_D w\, dA$, the mean temperature $t_m = (1/A) \int_D t\, dA$, and the mixed mean temperature $t_M = (1/A\, w_m) \int_D t\, w\, dA$ can be written in general by employing the complex Stokes' Theorem [6], e.g.

$$\langle w \rangle = (\pi \rho\, c_1/8) [\sum_0 b_r \beta_{-r} + \sum_1 \bar{b}_r \beta_r - 4 \sum_1 r a_r \bar{A}_r] \tag{20a}$$

where

$$\left.\begin{aligned} \beta_n &= \sum_0 (n + r)\, a_{n+r}\, \bar{a}_r; \quad \beta_{-n} = \sum_0 r\, a_r\, \bar{a}_{n+r} \\ A_n &= (1/n) \sum_1^n r\, a_r\, D_{n-r}; \quad D_0 = \tfrac{1}{2} b_0,\; D_n = b_n\; (n \geq 1). \end{aligned}\right\} \tag{20b}$$

From mass flow rate, the mass velocity G is obtainable as

$$G = \langle w \rangle / A.$$

After some manipulation it can be verified that if we had ignored the contribution of pressure drop in the energy equation, we would have got a different expression for temperature; likewise for mean and mixed-mean temperatures. These, again for an arbitrary B, are obtainable in general by making use of complex variable techniques. For the present we avoid this, but these will be given directly for some B's which have been chosen to be considered as illustrative examples in the following:

Example 1: Equilateral triangular duct

Let us consider the tube of an equilateral triangular cross-section. Let the sides be $2(3)^{\frac{1}{2}} a$, and the equations of the boundary be

$$x - a = 0; \quad x - (3)^{\frac{1}{2}} y + 2a = 0; \quad x + (3)^{\frac{1}{2}} y + 2a = 0. \tag{21}$$

Equations (21) are expressible in the form (13) as

$$Z\bar{Z} = (4/3)\,a^2 - [(1/6a)(Z^3 + \bar{Z}^3)]. \tag{22}$$

Avoiding the details of the calculations, the results for this duct are

$$\langle w \rangle = -\,(g\sqrt{3}/20)\,\rho\,c_1\,a^4,\quad G = -\,(3/20)\,c_1\,a^2\,\rho \qquad (23a,\ 23b)$$

$$\hat{t} = \hat{t}_w - (25/162)\,(x^+ - 1)^2\,[(x^+ + 2)^2 - 3y^{+2}]^2 \tag{24}$$

$$\hat{t} - \hat{t}_d = -\,(25/108)\,(x^+ - 1)[(x^+ + 2)^2 - 3y^{+2}]\,(x^{+2} + y^{+2} - 4) \tag{25}$$

$$\hat{t}_m = \hat{t}_w - (5/7),\quad \hat{t}_m - \hat{t}_{md} = -\,(10/7) \qquad (26a,\ 26b)$$

$$\hat{t}_M = \hat{t}_w - (90/77),\quad \hat{t}_M - \hat{t}_{Md} = -\,(15/7) \qquad (27a,\ 27b)$$

$$q = 0,\ q - q_d = 20\sqrt{3}(K\,Pr\,G^2/g\,c_p\,\rho^2) \qquad (28a,\ 28b)$$

where

$$\hat{t} = (t\,g\,c_p\,\rho^2/Pr\,G^2),\quad x^+ = (x/a),\quad y^+ = (y/a). \qquad (29a,\ 29b,\ 29c)$$

Example 2: Elliptic tube

In the case of an elliptic tube, for which the boundary equation

$$(x^2/a^2) + (y^2/b^2) = 1 \tag{30}$$

(a is semi-major axis, b is semi-minor axis)

is expressible as

$$Z\bar{Z} = \tfrac{1}{2}h_1(Z^2 + \bar{Z}^2) + 2h_2, \tag{31}$$

$$h_1 = (1 - \lambda^2)/(1 + \lambda^2),\ h_2 = a^2\lambda^2/(1 + \lambda^2),\quad \lambda = a/b, \qquad (32a,\ 32b,\ 32c)$$

the results are

$$\langle w \rangle = -\frac{\pi\,a^4\,\lambda^3}{4(1 + \lambda^2)}\,c_1\,\rho,\quad G = -\frac{a^2\,\lambda^2}{4(1 + \lambda^2)}\,c_1\,\rho \qquad (33a,\ 33b)$$

$$\hat{t} = \hat{t}_w - 2\,[x^{+2} + (y^{+2}/\lambda^2) - 1]^2 \tag{34}$$

$$\hat{t} - \hat{t}_d = -\frac{(1 + \lambda^2)}{2\lambda^2}\left(x^{+2} + \frac{y^{+2}}{\lambda^2} - 1\right)[(A_1 - 2A_2)x^{+2} + (A_1 + 2A_2)y^{+2} + 2(A_1 - 4)\,(h_2/a^2)] \tag{35}$$

$$\hat{t}_m = \hat{t}_w - \tfrac{2}{3},\quad \hat{t}_m - \hat{t}_{md} = -\,\tfrac{4}{3} \qquad (36a,\ 36b)$$

$$\hat{t}_M = \hat{t}_w - 1,\quad \hat{t}_M - \hat{t}_{Md} = -\frac{1}{9}\left(\frac{17 + 98\lambda^2 + 17\lambda^4}{1 + 6\lambda^2 + \lambda^4}\right) \qquad (37a,\ 37b)$$

$$q = 0,\quad q - q_d = 4\pi\left(\frac{1 + \lambda^2}{\lambda}\right)\frac{K\,Pr\,G^2}{g\,c_p\,\rho^2} \qquad (38a,\ 38b)$$

where

$$A_1 = \frac{2}{3}\left(\frac{1 + 10\lambda^2 + \lambda^4}{1 + 6\lambda^2 + \lambda^4}\right),\quad A_2 = \frac{1}{3}\left(\frac{1 - \lambda^4}{1 + 6\lambda^2 + \lambda^4}\right). \qquad (39a,\ 39b)$$

Examples 3, 4: Round and flat conduits

Setting $b = a$ (or $\lambda = 1$) in the results of Example 2, one can obtain the solution for the circular tube of radius a. Next, letting a approach infinity and b remain finite in the results of Example 2, one can obtain the results for the flat conduit with a gap b between walls.

In both cases results are in agreement with Madejski, we need not cite them here as they are available in reference [7].

Example 5: Cardioid duct

If the equation of the Cardioid boundary is

$$r^+ = 2(1 + \cos\sigma),\quad r^+ = r/l \qquad (40a,\ 40b)$$

then the conformal map is

$$Z = l\,(1 + \zeta)^2 \tag{41}$$

and as a consequence the results are

$$\langle w \rangle = -\frac{17}{4}\pi\,l^4 c_1\,\rho,\quad G = -\frac{17}{24}l^2 c_1\,\rho \qquad (42a,\ 42b)$$

$$\hat{t} = \hat{t}_w - \frac{18}{289}\left[r^{+2} - 4(r^+)^{\frac{1}{2}}\cos\frac{\sigma}{2} - 2r^+\cos\sigma\right]^2 \tag{43}$$

$$\hat{t} - \hat{t}_d = -\frac{3}{289}\Big[3r^{+4} + (168 - 32r^{+2})(r^+)^{\frac{1}{2}}\cos\frac{\sigma}{2} + 3r^+(28 - 4r^{+2})\cos\sigma + 32r^+(r^+)^{\frac{1}{2}}\cos\frac{3\sigma}{2} + 6r^{+2}\cos 2\sigma\Big] \tag{44}$$

$$\hat{t}_m = \hat{t}_w - \frac{194}{289}, \; \hat{t}_m - \hat{t}_{md} = -\frac{388}{289} \quad (45a, 45b)$$

$$\hat{t}_M = \hat{t}_w - \frac{3 \times 16638}{289 \times 170},$$

$$\hat{t}_M - \hat{t}_{Md} = -\frac{3 \times 30503}{289 \times 170} \quad (46a, 46b)$$

$$q = 0, \quad q - q_d = \frac{144\pi}{17} \cdot \frac{K\,Pr\,G^2}{g\,c_p\,\rho^2}. \quad (47a, 47b)$$

6. WALL TEMPERATURE DEVIATION AND TEMPERATURE DROP IN THE CHANNEL

From (15) it is obvious that the adiabatic conditions are fulfilled at the wall. Under such a condition, one has [7]

$$A w_m (c_p t_0 + w_m^2/2g) = \int_D w \; (c_p t + w^2/2g)\,\mathrm{d}A. \quad (48)$$

This, for constant property flow, gives

$$\Delta t_w^+ = K_2 - \frac{1}{Pr}(K_2 - K_1),$$

$$\Delta t_{\min}^+ = (1 - K_2) + \frac{1}{Pr}(K_2 - K_1). \quad (49a, 49b)$$

where $\Delta t_w^+ = t_w^+ - t_0^+$ is the deviation of wall temperature from the initial temperature (i.e. entrance temperature) t_0, $\Delta t_{\min}^+ = t_0^+ - t_{\min}^+$ is the temperature drop in the fully developed region in a given channel, and K_1, K_2, t^+ are defined as

$$K_1 = \frac{w_m^2}{w_{ex}^2}, \; K_2 = \frac{(w^3)_m}{w_m\,w_{ex}^2}, \qquad t^+ = \frac{2g\,c_p\,t}{Pr\,w_{ex}^2}. \quad (50a, 50b, 50c)$$

Obviously, the dimensionless constants K_1 and K_2 are different for different ducts, $K_2 > K_1$ and $K_2 < 1$.

For illustration, let us consider the following examples:

Example 1

In the case of an equilateral triangular duct (21), we find

$$K_1 = \frac{81}{400}, \quad K_2 = \frac{729}{1540} \quad (51a, 51b)$$

$$\left.\begin{aligned} \Delta t_w^+ &= \frac{729}{1540} - \frac{1}{Pr} \cdot \frac{8343}{30800}, \\ \Delta t_{\min}^+ &= \frac{811}{1540} + \frac{1}{Pr}\frac{8343}{30800}. \end{aligned}\right\} \quad (52a, 52b)$$

When $Pr = 0{\cdot}5722$ it is found that Δt_w becomes zero. For smaller Prandtl number this temperature difference becomes large and negative but for increasing Prandtl number it tends asymptotically to 0·4734. This is shown in Fig. 1.

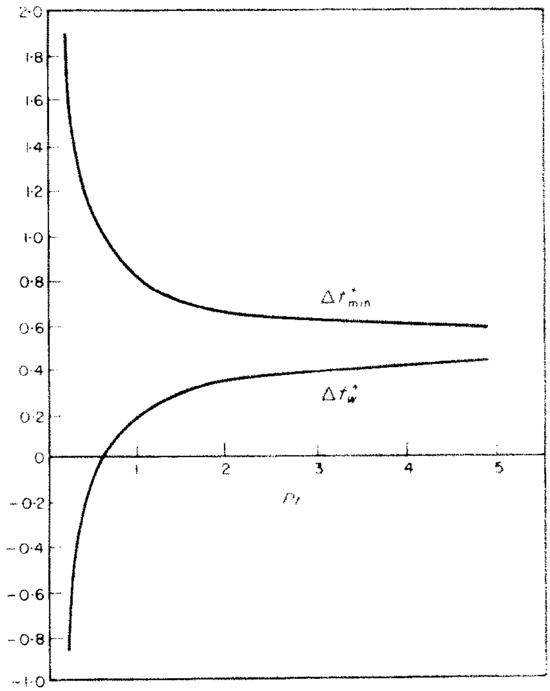

FIG. 1. Equilateral triangular duct. Δt^+_w and $\Delta t^+_{\min}$ vs. Prandtl number.

Example 2

In the case of an elliptic tube (30), the results are as follows:

$$K_1 = \tfrac{1}{4}, \quad K_2 = \tfrac{1}{2} \quad (53a, 53b)$$

$$\Delta t_w^+ = \frac{1}{2} - \frac{1}{Pr}\frac{1}{4}, \quad \Delta t_{\min}^+ = \frac{1}{2} + \frac{1}{Pr} \cdot \frac{1}{4}. \quad (54a, 54b)$$

These results are the same as have been obtained by Madejski [7] for round and flat conduits, further they are independent of the aspect ratio λ.

7. SKIN FRICTION

If n denotes the outward drawn normal through any current position on B, then from (6b) we have

$$\left(\mu \frac{\mathrm{d}w}{\mathrm{d}n}\right)_w = \left(\mu_w \frac{\mathrm{d}w_2}{\mathrm{d}n}\right)_w. \tag{55}$$

From this, it is quite clear that the skin friction in an arbitrary fully developed variable property flow may be determined from the constant property fluid analogue (8a). Thus from (14) we have for the special class of boundaries (13).

$$\hat{\tau}_l = \frac{1}{4}\left\{\cos\gamma\left[\left(\frac{Z}{L}+\frac{\bar{Z}}{L}\right) - \left(\frac{f'(Z)}{L}+\frac{\bar{f}'(\bar{Z})}{L}\right)\right] + \sin\gamma\left[\frac{1}{i}\left(\frac{Z}{L}-\frac{\bar{Z}}{L}\right)+\frac{1}{i}\left(\frac{f'(Z)}{L}-\frac{\bar{f}'(\bar{Z})}{L}\right)\right]\right\} \tag{56}$$

$$\hat{\tau}_m = A/LS, \tag{57}$$

and from (12) we have for an arbitrary channel

$$\hat{\tau}_l = \frac{1}{4L\left|\sum\limits_0 n\, a_n\, \xi^{n-1}\right|}\left(\sum_0 n\, a_n\, \xi^n \sum_0 \bar{a}_n\, \xi^{-n} + \sum_0 n\, \bar{a}_n\, \xi^{-n} \sum_0 a_n\, \xi^n - \sum_1 n\, b_n\, \xi^n - \sum_1 n\, \bar{b}_n\, \xi^{-n}\right) \tag{58}$$

$$\tau_m = \left(\pi \sum_1 j\, a_j\, \bar{a}_j\right)\Big/\left(L \int_{-\pi}^{\pi} \left|\sum_1 n\, a_n\, \xi^{n-1}\right| \mathrm{d}\theta\right). \tag{59}$$

Here L is some characteristic length in the cross section. As far as the graphical representation is concerned, it is better to picture the local skin friction and mean skin friction in the dimensionless forms $\hat{\tau}_l$ and $\hat{\tau}_m$ respectively, because the expressions of these [i.e. the right-hand side of (56) or that of (58), and the same of (57) or that of (59)] are independent of the variability of the fluid properties. Such graphical representation is shown in Figs. 2–5.

For engineering interest, it is desirable to evaluate skin friction in terms of mass velocity or Reynolds number. It is simple to do this in the case of constant property flow, the local and mean skin frictions in this case are given below.

(i) for an equilateral triangular tube (21):

$$\tau_l^+ = \tfrac{10}{3} R_G^{-1}\{\cos\gamma\,[x^+ + \tfrac{1}{2}(x^{+2} - y^{+2})] + \sin\gamma[\,y^+ - x^+ y^+]\} \tag{60}$$

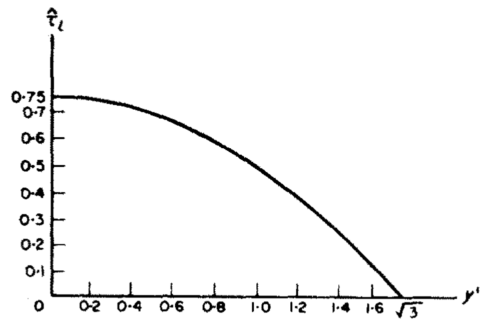

FIG. 2. Equilateral triangular duct.
Dimensionless skin-friction $\hat{\tau}_l = [\tau_l/(-a\, \mathrm{d}p/\mathrm{d}z)]$ vs. dimensionless distance $y'(=y/a)$. This figure shows the distribution of dimensionless skin-friction over any one of the sides from its centroid to its vertex $y' = 0$ corresponds to centroid and $y' = \sqrt{3}$ corresponds to vertex.

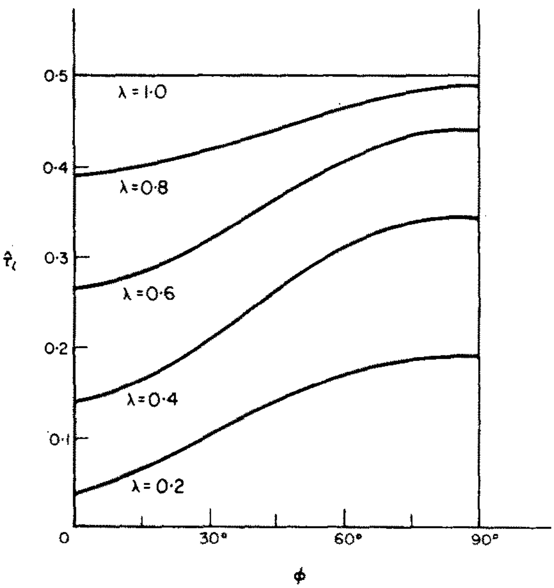

FIG. 3. Elliptic tube.
Dimensionless skin-friction $\hat{\tau}_l$ [$= \tau_l/(-a\, \mathrm{d}p/\mathrm{d}z)$] vs. eccentric angle ϕ with aspect ratio $\lambda\ (=b/a)$ as parameter.

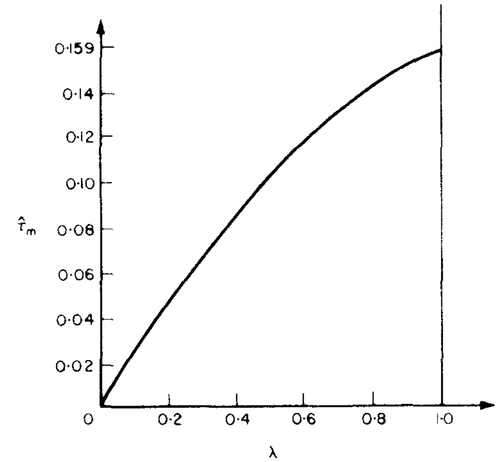

FIG. 4. Elliptic tube.
Dimensionless mean skin-friction $\hat{\tau}_m[=\tau_m/(-\pi a\,dp/dz)]$ vs. aspect ratio $\lambda(=b/a)$.

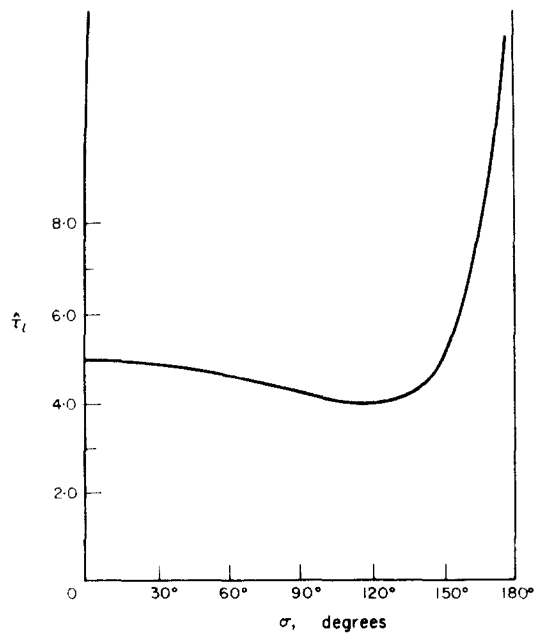

FIG. 5. Cardioid tube.
Dimensionless skin-friction $\hat{\tau}_l[=\tau_l/(-l\,dp/dz)]$ vs. vectorial angle σ. The minimum value of skin-friction corresponds to $\sigma = 120°$.

$$\tau_m^* = \frac{10}{3} R_G^{-1} \tag{61}$$

(ii) for an elliptic tube (30):

$$\tau_l^* = 4R_G^{-1}\frac{1}{\lambda}\sqrt{(\sin^2\phi + \lambda^2\cos^2\phi)} \tag{62}$$

$$\tau_m^* = R_G^{-1}\pi\frac{1+\lambda^2}{\lambda E[\sqrt{(1-\lambda^2)}]} \tag{63}$$

where $E(k)$ is the complete elliptical integral of the second kind, and (iii) for a Cardioid duct (40):

$$\tau_l^* = \frac{6}{17}R_G^{-1}\left[(2\cos\sigma + 3)/\cos\frac{\sigma}{2}\right] \tag{64}$$

$$\tau_m^* = \frac{9\pi}{17}R_G^{-1} \tag{65}$$

where the Reynolds number R_G is referred to the mass velocity and the characteristic length, i.e. a in the case of equilateral triangular tube (21), semi-major axis in the case of elliptic tube (30), and l in the case of Cardioid duct (40).

8. DISCUSSIONS AND CONCLUDING REMARKS

(i) From equations (5b), (6a) and (10) we find that the normal gradient of the temperature on the boundary is zero. Hence, the heat transfer between a solid boundary and fluid is zero, no matter how the fluid properties vary with temperature. If we had neglected the contribution of pressure drop, then we would have got non-zero heat transfer. This is confirmed from equations (28), (38) and (47).

(ii) From the discussion of Section 7, we conclude that the dimensionless skin friction for the variable property flow (in which fluid properties vary with temperature in an arbitrary manner) remains the same as that in the constant property flow.

(iii) Again, from equations (5b), (6a), and (10) we find that the temperature (whatever may be the manner in which fluid properties vary) decreases in the direction of the inward drawn normal and as a consequence the minimum temperature is attained at a point where the fluid velocity attains its maximum value. This result for constant property flows was known earlier [7].

(iv) From Section 5, it is clear that the velocity and temperature fields in the variable property flow are different from those in the constant property flow. In the special situation, where the ratio of viscosity and thermal conductivity is constant, the temperature is found to be a linear function of the square of velocity w. This is similar to what has been obtained by Madejski [7]; the only difference is that the velocity fields are different.

It is observed that the temperature field, obtained for the constant property flow by taking into account the compression work, is quite different from that obtained by ignoring it, e.g. in the case of an elliptic tube it is clear from equation (34) that if we take the family of similar and similarly situated ellipses,

$$x^{+2} + (y^{+2}/\lambda^2) - 1 = k$$

then these ellipses are equi-temperature and equi-velocity curves, where as this would have not been the case if we had ignored the effect of compression work which is clear from (34) and (35).

From equations (25), (35) and (44) it can be concluded that the local contribution of the pressure drop to the temperature field is quite significant, the average effect is given by (26b), (36b) and (45b) respectively for the cases of equilateral triangular, elliptic, and cardioid tubes. This is quite significant as it is numerically larger than the effect of viscous dissipation. The same conclusion is drawn about the mixed-mean effect of the pressure drop.

From equations (26), (36) and (45), it is interesting to note that

$$t_w = \tfrac{1}{2}(t_m + t_{md}).$$

However, from the illustrative examples, it is in general concluded that the effect of pressure drop is to decrease, and that of viscous dissipation is to increase the temperature away from the wall in the inward drawn normal direction. The effect of the former is numerically larger than that of the latter, the absolute difference is quite significant and reaches its maximum at the point where velocity attains its extreme value, and the two effects on the heat-transfer rate at the solid boundary are equal in magnitude and opposite in direction.

(v) In discussing Section 6 we conclude that the difference of wall temperature from the initial temperature and the temperature drop within the channel beyond the inlet length depend on the Prandtl number as well as on the configuration of the given channel. The dependence on Prandtl number had been pointed out earlier by Madejski [7] for circular and flat conduits only, while the effect of the configuration of the channel can be verified by comparing the results of equilateral triangular and elliptic tubes. However, for the ducts of similar and similarly situated cross-sections of the same kind they depend only on the Prandtl number. This is verified from the results of the case of elliptic tube, as (54a) and (54b) do not involve the aspect ratio λ.

ACKNOWLEDGEMENTS

The author is greatly indebted to Prof. J. N. Kapur, Head of the Mathematics Department, Indian Institute of Technology, Kanpur, for his kind guidance and constant attention during the preparation of this paper. He is also very grateful to three referees for their valuable suggestions.

REFERENCES

1. R. B. Bird, W. E. Steward and E. N. Lightfoot, *Transport Phenomena*. John Wiley, New York (1962).
2. M. V. Dyke, Higher approximation in boundary layer theory (Part I), *J. Fluid. Mech.* **14**, 2 (1962).
3. S. Goldstein, *Lectures on Fluid Mechanics*. Interscience Publishers, New York (1960).
4. L. V. Kantorovich and V. I. Krylov, *Approximate Methods of Higher Analysis*. Interscience Publishers, New York (1958).
5. M. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*. P. Noordhoff, Gröningen, Holland (1953).
6. L. M. Milne-Thomson, *The Theoretical Hydrodynamics* (Fourth ed.). Macmillan, London (1960).
7. J. Madejski, Temperature distribution in channel

flow with friction, *Int. J. Heat Mass Transfer* **6**, 49–51 (1963).
8. L. ROSENHEAD, *Laminar Boundary Layers*. Clarendon Press, Oxford (1963).
9. N. RILEY, The thermal boundary layer in the flow between converging plane walls, *Proc. Camb. Phil. Soc.* **59**, 225–229 (1963).
10. I. S. SOKOLNIKOFF, *Mathematical Theory of Elasticity*. McGraw-Hill, New York (1956).
11. H. SCHLICHTING, *Boundary Layer Theory*. McGraw-Hill, New York (1962).
12. E. M. SPARROW and J. L. GREGG, The variable fluid property problem in free convection, *Trans. Amer. Soc. Mech. Engrs* **80**, 879–886 (1958).

Résumé—Cet article discute le problème de la convection forcée avec un écoulement laminaire permanent entièrement développé dans un canal rectiligne lorsque les propriétés du fluide sont fonctions de la température. L'équation de l'énergie a été analysée en tenant compte de l'effet du travail de compression aussi bien que de celui de la dissipation visqueuse. La technique de la variable complexe est employée pour résoudre le cas d'un canal arbitraire. Les caractéristiques des écoulements à propriétés constantes et à propriétés variables ont été comparés. Comme illustration, quelques écoulements à propriétés constantes ont été obtenus analytiquement et les résultats lorsqu'on tient compte de l'effet du travail de compression ont été comparés avec ceux obtenus en le négligeant. Pour une section droite de forme arbitraire, les expressions donnant la différence entre la tempéature pariétale et la température initiale ainsi que la chute detempérature dans le tuyau ont été obtenus. Dans le paragraphe de la conclusion, les résultats ont été discutés.

Zusammenfassung—Diese Arbeit behandelt das Problem der Zwangskonvektion bei stationärer, laminarer, voll ausgebildeter Strömung in einem geraden Kanal, wenn die Stoffgrössen der Flüssigkeit Funktionen der Temperatur sind. Die Energiegleichung wurde unter Einbeziehung des Einflusses sowohl der Kompressionsarbeit wie auch der Reibungswärme analysiert. Die Methode der komplexen Variablen wird verwendet, um den Fall eines beliebigen Kanals zu behandeln. Die Strömungscharakteristiken für konstante und variable Stoffeerte wurden verglichen. Zur Illustration wurden einige Strömungen mit konstanten Stoffwerten in einer geschlossenen Form berechnet und die Ergebnisse, die sich durch Einbeziehen der Auswirkung der Kompressionsarbeit ergaben, wurden mit den Ergebnissen, die man durch ihre Vernachlässigung erhält, verglichen. Für einen beliebigen Querschnitt wurden die Ausdrücke, welche den Unterschied der Wandtemperatur von der Anfangstemperatur und das Temperaturgefälle im Kanal angeben, abgeleitet. Im anschliessenden Kapitel wurden die Ergebnisse diskutiert.

Аннотация—В статье рассматривается задача вынужденной конвекции при стационарном ламинарном полностью развитом течении в прямом канале жидкости с зависящими от температуры свойствами. Дан анализ уравнения энергии с учётом влияния работы сжатия, а также вязкостной диссипации. Для рассмотрения случая канала произвольного сечения использован метод комплексных переменных. Сравнены характеристики потока с постоянными свойствами и потока с переменными свойствами. В качестве иллюстрации получены решения в замкнутой форме для некоторых потоков с постоянными свойствами, и результаты, исследованные с учётом влияния работы сжатия, сравнены с результатами, полученными, когда таким влиянием пренебрегали. Для канала с произвольным поперечным сечением выведены выражения для разности между температурой стенки и начальной температурой и падения температуры в канале. В заключительной части проводится анализ результатов.